Lecture 3: Basic Counting and Conditional Probability

- Finite Sample Spaces
- Counting Methods
- Combinatorial Methods
- Conditional Probability

Introduction

– In the last lecture we discussed the different interpretations of probability.

– We also talked about the mathematical definition of probability and how to calculate probability of events in simple experiments.

– Today, we will discuss how finding the probability of an event will be reduced to counting the elements of the sample space that satisfy some condition.

– Moreover, how to calculate the probability of one event given what we know about another event that has already occurred.

– This would lead us to the concept of "independent events"

Finite Sample Spaces

– **Finite Sample Space** involves only finite many possible outcomes

– **Simple Sample Space** is a finite sample space such that every outcome in S has the same probability. If there are n outcomes in a simple sample space S, then each on must have probability $1/n$.

– Revisit the "Roll a die twice" Exercise

– Compute the probability of an event as

$$\frac{\text{Number of favorable cases}}{\text{Number of possible outcomes}}$$

– **Enumerating all possible outcomes can be inefficient and error-prone**

Multiplication Rule

– First we need to review important methods for counting the number of outcomes in an event/set.

– This is particularly useful when the process of enumerating all possible outcomes is simply inefficient.

– **Multiplication Rule (D.S. Theorem 1.7.2)**: Suppose an experiment has k parts $(k \geq 2)$, such that the ith part of the experiment can have n_i possible outcomes $(i = 1, \ldots, k)$. If all of the outcomes in each part can occur regardless of which specific outcomes have occurred in the other parts, then the total number of outcomes will be equal to the product: $n_1 n_2 ... n_k$.

Examples

– *Example 1:* Back to our toy example of tossing a coin twice. Since the outcome of each toss doesn't affect the other, the total number of outcomes, that is $|\Omega|$, is $2 \times 2 = 4$

– *Example 2:* Suppose that I have 5 T-shirts and 3 pairs of shorts. In total, I have $5 \times 3 = 15$ distinct "outfits"

– *Example 3:* Suppose your favorite frozen yogurt place has 4 different choices of yogurt and 10 different toppings. There are $4 \times 10 = 40$ different 1-topping choices

Permutations

– **Permutations**: The number of ways to arrange n distinct objects in a line is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

This is the number of permutations of n distinct objects. There are n choices for the the first position, then $n-1$ for the second and so forth; multiplication gives the number of distinct elements.

– Then the number of permutations of n elements taken k at a time is

$$P_{n,k} = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

– By convention, 0! = 1. Thus, 1! = 1, 2! = 2, 3! = 6, 4! = 24, ...

– $P_{n,k}$ is also the number of distinct "orderings" of k items selected without replacement from a collection of n different items.

Examples

– *Example 3:* In how many ways can 8 people line up? $8! = 40320$.

– *Example 4:* In how many ways can 4 married couples stand in a police line-up , if couples must stand together?

There are $4! = 24$ ways that the couples can be arranged, and each couple can be arranged in $2! = 2$ ways. Thus the answer is

$$4! \times 2! \times 2! \times 2! \times 2! = 384$$

– *Example 5 (D.S. Example 1.7.8):* In how many ways can you select a president and a secretary from a club consisting of 25 members?

Combinations

– Notice that ordering is important in permutations. What if we wish to select k items from a collection of n different items without regard for specific orderings? In other words, we care about selecting subsets of size k.

– **Combinations**: The number of distinct subsets of size k that can be chosen from a set of size n is:

$$C_{n,k} = \binom{n}{k} = \frac{P_{n,k}}{k!} = \frac{n!}{k!(n-k)!}$$

– That is, once we have the number of possible permutations, we need to get rid of repeated subsets. Since there are $k!$ ways to arrange each of the k repeated elements, we divide by $k!$.

– **$\binom{n}{k}$ is also called the binomial coefficient but we will get to that later in the course.**

Examples

– *Example 6 (D.S. Example 1.8.2):* The number of different groups of people that might be on the committee composed of eight people from a group of 20 people is?

– *Example 7:* What is the probability of a four of a kind in 5 card poker? We need to compute

$$P(4 \text{ of a kind}) = \frac{\#\,(\text{favorable hands})}{\#\,(5 \text{ card hands})}$$

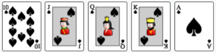
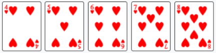
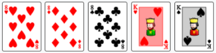
where

$$\#\,(5 \text{ card hands}) = \binom{52}{5}, \quad \#\,(\text{favorable hands}) = 13 \times (52 - 4) = 624$$

because we can (1) pick any rank from $A - K$ (13 choices), (2) choose any other card (we have 52 minus the 4 that we've taken)

Poker Probabilities

The *nCr* function on most scientific calculators can be used to calculate hand frequencies; entering `nCr` with `52` and `5`, for example, yields $\binom{52}{5} = 2,598,960$ as above.

Hand	Distinct hands	Frequency	Probability	Cumulative probability	Odds	Mathematical expression of absolute frequency
Royal flush 	1	4	0.000154%	0.000154%	649,739 : 1	$\binom{4}{1}$
Straight flush (excluding royal flush) 	9	36	0.00139%	0.0015%	72,192 : 1	$\binom{10}{1}\binom{4}{1} - \binom{4}{1}$
Four of a kind 	156	624	0.0240%	0.0256%	4,164 : 1	$\binom{13}{1}\binom{12}{1}\binom{4}{1}$
Full house 	156	3,744	0.1441%	0.17%	693 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$

Wikipedia article on poker probability: `https://en.wikipedia.org/wiki/Poker_probability`

Conditional Probability

– In the previous classes, we saw how to calculate the probability of events in a simple experiment. We also saw how to calculate the probability that two events occur together. That is, for two events A and B, P(A and B) = P(A ∩ B).

– What if we are instead interested in the probability that A happens given that B has already happened? This is called **conditional probability**. Some examples to illustrate this:

If $A = \{\text{a person is 6 ft or taller}\}$ and $B = \{\text{a person is a basketball player}\}$, $P(A|B)$ is the probability that a person is 6 ft or taller given that she is a basketball player. Is $P(A|B)$ greater or less than $P(A)$?

Example 8

Example 8: Back to our toy example of tossing a fair coin twice. Let A be the event that only one head is observed, and B the event that at least one head is observed. Then A $= \{HT, TH\}$ and B = $\{HH, HT, TH\}$. What is the probability that A occurs given that we know B already occurred?

First notice that if B already happened, $P(\{TT\}) = 0$, so that B is our new sample space. If B is the new sample space, and all outcomes are equally likely, then the probability we seek to calculate is simply $\frac{2}{3}$.

Definition

– Formally, the probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

which is not defined if $P(B) = 0$. That is, it doesn't make sense to condition on an event that cannot happen.

– The interpretation is "the probability that A happens given that we know that B has happened".

– $P(A)$ is sometimes called the marginal probability of A. (In contrast with the conditional probability of A, given B)

– $P(A|B) + P(A^c|B) = 1$. (Conditional probabilities behave just like probabilities.)

Examples

Example 8 again: Using this formula instead of what we did before,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/4}{3/4} = \frac{2}{3} \quad \text{same as before!}$$

– A useful implication of conditional probability is:

$$P(A \cap B) = P(A|B)P(B)$$

$$\Rightarrow P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$$

Probability Update

– Suppose that our friend Bobby throws a die. He knows that outcome, but we don't. He tells us that the result is an even number. What is the probability that it is a 6?

– Well, our intuition tells us that it should be $1/3$, since the only possible outcomes are 2, 4, and 6, and they should be equally likely.

– On the other hand, the probability that the result is, say a 5, is 0, because we know that the outcome is an even number.

– Therefore, $P(5) = P(6) = 1/6$, but $P(5|B) = 0$ and $P(6|B) = 1/3$, where B is the information that Bobby gave us.

– Apply the mathematical definition: $P(B) = 1/2$, $P(6 \cap B) = P(6) = 1/6$, $P(5 \cap B) = P(\emptyset) = 0$. Then $P(6|B) = 1/3$ and $P(5|B) = 0$.

More Examples

Example 9: Suppose you have a standard deck of 52 cards which includes 13 cards of each of the four suits: clubs, diamonds, hearts and spades. Clubs and spades are black while diamonds and hearts are red. Let A be the event that you draw a diamond card and B the event that you draw a red card. Then,

- $P(A) = \frac{13}{52} = \frac{1}{4}$
- $P(B) = \frac{26}{52} = \frac{1}{2}$
- $P(A \text{ and } B) = P(A \cap B) = \frac{13}{52} = \frac{1}{4}$
- $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$
- $P(A|B) = \frac{1/4}{1/2} = \frac{1}{2}$
- $P(B|A) = \frac{1/4}{1/4} = 1$. So if A already happened, B has to happen.

Excercise

For each of the following events A and B, do you think that $P(A|B)$ less than, equal to, or greater than $P(A)$?

1. A : high temperature over 100 degrees, B: summer
2. A: a card drawn from a well-shuffled deck is greater than 5, B: the card is a club
3. A: understanding conditional probability, B: having done this exercise

Come up with more examples of your own.

Independent Events

– Two events A and B are said to be independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Notice already that if A and B are independent,

$$P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A), \quad P(B|A) = \frac{P(A)P(B)}{P(A)} = P(B)$$

so that A and B are independent if the occurrence of one doesn't affect the other in any way.

– **Can you see why this is true for our toy example of tossing a fair coin twice? Define any event based on the first toss and another event based on the second toss. Then use conditional probability to show independence between them.**

Examples

– *Example 9 again:* Are A and B independent?

$$P(A \cap B) = \frac{1}{4} \neq \frac{1}{4} \times \frac{1}{2} = P(A)P(B)$$

Of course not! If I already picked a red card, I know the card has to be a diamond or a heart, which changes my uncertainty about picking a diamond card.

– *Example 10 (D.S. Example 2.2.3) :* Suppose that a balanced/fair die is rolled. Let A be the event that an even number is obtained, and let B be the event that one of the numbers 1, 2, 3, or 4 is obtained. Are they independent?

– These examples help present the idea of independence as information gain. If I know an event A has happened, do I gain any information from A about another event B beyond what I already know about B? If no, they are independent.

Recap

We discussed the following:

- Some useful counting methods
- Conditional probability
- The concept of independent events

We'll continue talking about independent events in the next lecture.

Suggested reading:

- D.S. Sec. 1.7, 1.8, 2.1, 2.2
- OpenIntro3: Sec. 2.2.1-2.2.5